Arab J Math (2014) 3:325–340
DOI 10.1007/s40065-014-0103-8

Belkacem Chaouchi · Fatiha Boutaous

An abstract approach for the study of an elliptic problem in a nonsmooth cylinder

Received: 13 December 2013 / Accepted: 11 May 2014 / Published online: 4 July 2014

Abstract The aim of this work is the resolution of a non-autonomous abstract differential equation of elliptic type set on unbounded domain. The study is performed in the framework of Hölder spaces. An example for a concrete elliptic problem in nonsmooth cylindrical domains will illustrate the theory.

Mathematics Subject Classification 34G10 · 34K10 · 12H2O

الملخص

هدف هذا العمل هو حل معادلة تفاضلية مجردة غير ذاتية من النوع الإهليليجي على مجال غير محدَّد.تتم هذه الدراسة في إطار فضاءات هولدر. يتم ترسيخ النظرية عبر مثال لمسألة إهليليجية مميَّزَة في مجالات أسطوانية غير ملساء.

1 Introduction

In this work, we deal with non-autonomous problems of the form

$$u''(t) + A(t)u(t) - \lambda u(t) = f(t), \qquad t \geq 0,\ \lambda > 0, \tag{1}$$

subject to the following boundary conditions

$$u(0) = 0,\ \ u(+\infty) = 0, \tag{2}$$

where:

(i) $f \in BUC^{2\sigma}([0, +\infty[; E), \sigma \in]0, 1/2[$, denoting the space of bounded and $2\sigma-$Hölder continuous functions $f : [0, +\infty[\to E$, endowed with the norm

$$\|f\|_{BUC^{2\sigma}([0,+\infty[;E)} = \sup_{t\geq 0} \|f(t)\|_E + \sup_{t\neq\tau} \frac{\|f(t) - f(\tau)\|_E}{|t-\tau|^{2\sigma}},$$

where E is a complex Banach space.

B. Chaouchi (✉)
Lab. de l'Energie et des Systémes Intelligents, Khemis Miliana University, 44225 Khemis Miliana, Algeria
E-mail: chaouchicukm@gmail.com

F. Boutaous
Département de Mathématiques, Faculté des Sciences, Sâad Dahlab University, B.P. 270, Blida, Algeria
E-mail: boutaous.fatiha2009@hotmail.com

(ii) $(A(t))_{t\geq 0}$ is a family of closed linear operators satisfying certain assumptions to be specified later on.

The aim of the present paper is twofold:

1. Give a complete study of Problems (1) and (2). We will then establish existence, uniqueness of the strict solution, that means a function u such that

$$\begin{cases} u \in BUC^2([0,+\infty[, E), \\ \text{for every } t \geq 0 : u(t) \in D(A(t)) \\ \text{and } (A(t)-\lambda)u(t) \in BUC([0,+\infty[; E), \end{cases}$$

and satisfying (1) and (2). Just, we recall here that for $k \in N$, $BUC^k([0,+\infty[; E)$ is the space of vector-valued functions with uniformly continuous and bounded derivatives up to order k in $[0,+\infty[$.
2. Exploit and apply the above results to establish some Hölder continuous regularity results for a concrete boundary value problem set on a singular domain.

It should be noted that the solvability of boundary value problems for differential operator equations on bounded domains has been widely studied. For an overview on these kind of problems and some historical references, see [11,15,16]. Several methods have been developed for the solution of this kind of problems. Among these methods, we cite as an example the semigroup techniques and the well-known sum's operators theory, see [6]. In this work, our strategy is based essentially on the use of the Dunford's integrals as in [3] and the methods applied in [1] and [12]. Note that besides being complementary to [5], the present paper contains important new results. In fact, we present some new Hölder continuous regularity results for an elliptic equation set on nonsmooth cylindrical domains. These results can be hardly obtained using the classical standard techniques such as the classical variational methods or the potential theory. This paper is organized as follows. In Sect. 2, we build the natural representation of the solution of (1) and (2) using the Dunford operational calculus. We prove also some results, which allow us to justify the optimal smoothness of the previous representation. In Sect. 3, we give a concrete example to which our abstract results can be apply.

2 Optimal results for Problems (1) and (2)

2.1 Assumptions and representation of the solution

For simplicity of notation, we set

$$Q(t) = A(t) - \lambda,\ \ \lambda > 0.$$

Throughout this work, we assume that the the family of linear closed operators $(Q(t), D(Q(t)))_{t\geq 0}$ enjoys the following properties:

1. $\forall t \geq 0, \exists\, \delta_0 \in]\frac{\pi}{2}, \pi[$such that

$$\rho(Q(t)) \supset \sum_{\delta_0} = \{z \in C : re^{i\delta}, r \in [0,+\infty[, \delta \in]-\delta_0, +\delta_0[\}, \tag{3}$$

here, $\rho(Q(t))$ is the resolvent set of $Q(t)$.
2. $\exists C_1 > 0 : \forall z \in \sum_{\delta_0}, \forall t \geq 0$

$$\|(Q(t) - zI)^{-1}\|_{L(E)} \leq \frac{C_1}{|z|+1}. \tag{4}$$

3. For all $z \in \sum_{\delta_0}$, the mapping $t \mapsto (Q(t)-zI)^{-1}$ defined on R^+, is of class C^2. Furthermore, we suppose also that:
There exist $C_2 > 0$, $\sigma \in]0, 1/2[$ such that for all $z \in \sum_{\delta_0}$ and all $t, \tau \geq 0$,

$$\begin{cases} \|\frac{\partial}{\partial t}(Q(t)-zI)^{-1}\|_{L(E)} \leq \frac{C_2}{|z|+1}, \\ \|\frac{\partial^2}{\partial t^2}(Q(t)-zI)^{-1}\|_{L(E)} \leq \frac{C_2}{|z|+1}, \end{cases} \tag{5}$$

and

$$\left\| \frac{\partial}{\partial t}(Q(t) - zI)^{-1} - \frac{\partial}{\partial \tau}(Q(\tau) - zI)^{-1} \right\|_{L(E)} \leq \frac{C_2 |t-\tau|^{2\sigma}}{|z|+1},$$
$$\left\| \frac{\partial^2}{\partial t^2}(Q(t) - zI)^{-1} - \frac{\partial^2}{\partial \tau^2}(Q(\tau) - zI)^{-1} \right\|_{L(E)} \leq \frac{C_2 |t-\tau|^{2\sigma}}{|z|+1}. \tag{6}$$

Remark 2.1 The above hypotheses are known in the literature as the Da Prato–Grisvard hypotheses, see [6] and [12]. Just, we note that:

1. All the constants given above are independent of t.
2. Hypotheses (3) and (4) express the ellipticity of (1). Moreover, all the previous assumptions remain true if we replace z by $z + \sqrt{\lambda}$.

Throughout the rest of this paper, C stands for a generic constant and $\sigma \in]0, 1/2[$,
We know that in the case when

$$Q(t) = Q,$$

is a constant operator satisfying the hypothesis (3) and (4), the representation of the solution u of (1) and (2) is given by

$$u(t) = -\frac{1}{2i\pi} \int_{\gamma} \int_{0}^{+\infty} k_{\sqrt{-z}}(t,s)(Q - z)^{-1} f(s) \mathrm{d}s, \tag{7}$$

where

$$k_{\sqrt{-z}}(t,s) = \begin{cases} \dfrac{\mathrm{e}^{-\sqrt{-z}t} \sinh \sqrt{-z}s}{\sqrt{-z}} & 0 \leq s \leq t, \\ \dfrac{\mathrm{e}^{-\sqrt{-z}s} \sinh \sqrt{-z}t}{\sqrt{-z}} & s \geq t. \end{cases} \tag{8}$$

Here, the curve γ is the boundary of the sector $\sum_{\delta_0}$ oriented from $\infty \mathrm{e}^{+i\delta_0}$ to $\infty \mathrm{e}^{-i\delta_0}$ and $\sqrt{-z}$ is the analytic determination defined by $\Re\sqrt{-z} > 0$, see [3].
In our situation, our representation formula can be heuristically derived by the following argument:
Taking the constant case into account, we look for a solution of Problems (1) and (2) in the following form

$$u(t) = -\frac{1}{2i\pi} \int_{\gamma} \int_{0}^{+\infty} k_{\sqrt{-z}}(t,s)(Q(t) - z)^{-1} f^*(s) \mathrm{d}s. \tag{9}$$

We are then concerned with the determination of the unknown function f^* in order that (9) is a strict solution of Problems (1) and (2).

2.2 Study of the regularity of the formal solution (9)

First, one has

Proposition 2.2 *Assume that $f^* \in BUC^{2\sigma}([0, +\infty[; E)$. Then,*

1. *For all $t \geq 0$, one has*

$$u(t) \in D(A(t)).$$

2. *For all $t \geq 0$, the function f^* (introduced in 9) satisfies the following equation*

$$f^*(t) - Op(f^*)(t) = f(t), \tag{10}$$

where

$$\begin{aligned}
Op(f^*)(t) = &+\frac{1}{2i\pi}\int_\gamma\left(\int_0^t \frac{e^{-\sqrt{-z}t}\sinh\sqrt{-z}s}{\sqrt{-z}}\frac{\partial^2}{\partial t^2}(Q(t)-zI)^{-1}f^*(s)ds\right)dz\\
&+\frac{1}{2i\pi}\int_\gamma\left(\int_t^{+\infty} \frac{e^{-\sqrt{-z}s}\sinh\sqrt{-z}t}{\sqrt{-z}}\frac{\partial^2}{\partial t^2}(Q(t)-zI)^{-1}f^*(s)ds\right)dz\\
&-\frac{1}{i\pi}\int_\gamma\left(\int_0^t e^{-\sqrt{-z}t}\sinh\sqrt{-z}s\frac{\partial}{\partial t}(Q(t)-zI)^{-1}f^*(s)ds\right)dz\\
&+\frac{1}{i\pi}\int_\gamma\left(\int_t^{+\infty} e^{-\sqrt{-z}s}\cosh\sqrt{-z}t\frac{\partial}{\partial t}(Q(t)-zI)^{-1}f^*(s)ds\right)dz.
\end{aligned}$$

Proof Statement 1 follows from Proposition 3.1 in [3].
Concerning Statement 2, one has

$$\begin{aligned}
u'(t) = &-\frac{1}{2i\pi}\int_\gamma\left(\int_0^t \frac{e^{-\sqrt{-z}t}\sinh\sqrt{-z}s}{\sqrt{-z}}\frac{\partial}{\partial t}(Q(t)-zI)^{-1}f^*(s)ds\right)dz\\
&-\frac{1}{2i\pi}\int_\gamma\left(\int_t^{+\infty} \frac{e^{-\sqrt{-z}s}\sinh\sqrt{-z}t}{\sqrt{-z}}\frac{\partial}{\partial t}(Q(t)-zI)^{-1}f^*(s)ds\right)dz\\
&-\frac{1}{2i\pi}\int_\gamma \frac{e^{-\sqrt{-z}t}\sinh\sqrt{-z}t}{\sqrt{-z}}(Q(t)-z)^{-1}f^*(t)dz\\
&+\frac{1}{2i\pi}\int_\gamma\left(\int_0^t e^{-\sqrt{-z}t}\sinh\sqrt{-z}s(Q(t)-zI)^{-1}f^*(s)ds\right)dz\\
&+\frac{1}{2i\pi}\int_\gamma \frac{e^{-\sqrt{-z}t}\sinh\sqrt{-z}t}{\sqrt{-z}}(Q(t)-z)^{-1}f^*(t)dz\\
&-\frac{1}{2i\pi}\int_\gamma\left(\int_t^{+\infty} e^{-\sqrt{-z}s}\cosh\sqrt{-z}t(Q(t)-zI)^{-1}f^*(s)ds\right)dz.
\end{aligned}$$

To calculate $u''(t)$, we follow the same reasoning as in [12]. We set

$$\begin{aligned}
&u'_\varepsilon(t)\\
&= -\frac{1}{2i\pi}\int_\gamma\left(\int_0^{t-\varepsilon} k_{\sqrt{-z}}(t,s)\frac{\partial}{\partial t}(Q(t)-zI)^{-1}f^*(s)ds\right)dz\\
&\quad-\frac{1}{2i\pi}\int_\gamma\left(\int_{t+\varepsilon}^{+\infty} k_{\sqrt{-z}}(t,s)\frac{\partial}{\partial t}(Q(t)-zI)^{-1}f^*(s)ds\right)dz
\end{aligned}$$

$$+\frac{1}{2i\pi}\int_{\gamma}\left(\int_{0}^{t-\varepsilon}\mathrm{e}^{-\sqrt{-z}t}\sinh\sqrt{-z}s(Q(t)-zI)^{-1}f^{*}(s)\mathrm{d}s\right)\mathrm{d}z$$
$$-\frac{1}{2i\pi}\int_{\gamma}\left(\int_{t+\varepsilon}^{+\infty}\mathrm{e}^{-\sqrt{-z}s}\cosh\sqrt{-z}t(Q(t)-zI)^{-1}f^{*}(s)\mathrm{d}s\right)\mathrm{d}z.$$

First, it is clear that

$$\lim_{\varepsilon\to 0}u'_{\varepsilon}(t)=u'(t).$$

On the other hand, one has

$$u''_{\varepsilon}(t) = R_{(\varepsilon)}(f^{*})(t)+I_{(\varepsilon)}(f^{*})(t)-Op_{(\varepsilon)}(f^{*})(t),$$

where

$$R_{(\varepsilon)}(f^{*})(t) = -\frac{1}{2i\pi}\int_{\gamma}\frac{\mathrm{e}^{-\sqrt{-z}t}\sinh\sqrt{-z}(t-\varepsilon)}{\sqrt{-z}}\frac{\partial}{\partial t}(Q(t)-zI)^{-1}f^{*}(t-\varepsilon)\mathrm{d}z$$
$$+\frac{1}{2i\pi}\int_{\gamma}\frac{\mathrm{e}^{-\sqrt{-z}(t+\varepsilon)}\sinh\sqrt{-z}t}{\sqrt{-z}}\frac{\partial}{\partial t}(Q(t)-zI)^{-1}f^{*}(t+\varepsilon)\mathrm{d}z,$$

and

$$I_{(\varepsilon)}(f^{*})(t) = +\frac{1}{2i\pi}\int_{\gamma}\mathrm{e}^{-\sqrt{-z}t}\sinh\sqrt{-z}(t-\varepsilon)(Q(t)-zI)^{-1}f^{*}(t-\varepsilon)\mathrm{d}z$$
$$+\frac{1}{2i\pi}\int_{\gamma}\mathrm{e}^{-\sqrt{-z}(t+\varepsilon)}\cosh\sqrt{-z}t(Q(t)-zI)^{-1}f^{*}(t+\varepsilon)\mathrm{d}z$$
$$+\frac{1}{2i\pi}\int_{\gamma}(-\sqrt{-z})\left(\int_{0}^{t-\varepsilon}\mathrm{e}^{-\sqrt{-z}t}\sinh\sqrt{-z}s(Q(t)-zI)^{-1}f^{*}(s)\mathrm{d}s\right)\mathrm{d}z$$
$$+\frac{1}{2i\pi}\int_{\gamma}(-\sqrt{-z})\left(\int_{t+\varepsilon}^{+\infty}\mathrm{e}^{-\sqrt{-z}s}\sinh\sqrt{-z}t(Q(t)-zI)^{-1}f^{*}(s)\mathrm{d}s\right)\mathrm{d}z,$$

and

$$Op_{(\varepsilon)}(f^{*})(t) = +\frac{1}{2i\pi}\int_{\gamma}\left(\int_{0}^{t-\varepsilon}\frac{\mathrm{e}^{-\sqrt{-z}t}\sinh\sqrt{-z}s}{\sqrt{-z}}\frac{\partial^{2}}{\partial t^{2}}(Q(t)-zI)^{-1}f^{*}(s)\mathrm{d}s\right)\mathrm{d}z$$
$$+\frac{1}{2i\pi}\int_{\gamma}\left(\int_{t+\varepsilon}^{+\infty}\frac{\mathrm{e}^{-\sqrt{-z}s}\sinh\sqrt{-z}t}{\sqrt{-z}}\frac{\partial^{2}}{\partial t^{2}}(Q(t)-zI)^{-1}f^{*}(s)\mathrm{d}s\right)\mathrm{d}z$$
$$-\frac{1}{i\pi}\int_{\gamma}\left(\int_{0}^{t-\varepsilon}\mathrm{e}^{-\sqrt{-z}t}\sinh\sqrt{-z}s\frac{\partial}{\partial t}(Q(t)-zI)^{-1}f^{*}(s)\mathrm{d}s\right)\mathrm{d}z$$

$$+\frac{1}{i\pi}\int_{\gamma}\left(\int_{t+\varepsilon}^{+\infty}\mathrm{e}^{-\sqrt{-z}s}\cosh\sqrt{-z}t\frac{\partial}{\partial t}(Q(t)-zI)^{-1}f^{*}(s)\mathrm{d}s\right)\mathrm{d}z.$$

For $R_{(\varepsilon)}(f^{*})(t)$, observe that

$$\begin{aligned}&\left\|\frac{\mathrm{e}^{-\sqrt{-z}t}\sinh\sqrt{-z}(t-\varepsilon)}{\sqrt{-z}}\frac{\partial}{\partial t}(Q(t)-zI)^{-1}f^{*}(t-\varepsilon)\right\|_{L(E)}\\&\quad\leq\frac{\mathrm{e}^{-\Re\sqrt{-z}t}\mathrm{e}^{\Re\sqrt{-z}(t-\varepsilon)}}{|z|^{1/2}|z|}\|f^{*}\|_{BUC^{2\sigma}([0,+\infty[,E)}\end{aligned}$$

and

$$\begin{aligned}&\left\|\frac{\mathrm{e}^{-\sqrt{-z}(t+\varepsilon)}\sinh\sqrt{-z}t}{\sqrt{-z}}\frac{\partial}{\partial t}(Q(t)-zI)^{-1}f^{*}(t+\varepsilon)\right\|_{L(E)}\\&\quad\leq\frac{\mathrm{e}^{-\Re\sqrt{-z}(t+\varepsilon)}\mathrm{e}^{-\Re\sqrt{-z}t}}{|z|^{1/2}|z|}\|f^{*}\|_{BUC^{2\sigma}([0,+\infty[,E)},\end{aligned}$$

by the dominated convergence theorem, we get

$$\lim_{\varepsilon\to 0}R_{(\varepsilon)}(f^{*})(.)=0.$$

For the quantity $Op_{(\varepsilon)}(f^{*})(t)$, we see that

$$\lim_{\varepsilon\to 0}Op_{(\varepsilon)}(f^{*})(.)=Op(f^{*})(.).$$

It remains to treat the quantity $I_{(\varepsilon)}(f^{*})(t)$, using the identity

$$Q(t)(Q(t)-zI)^{-1}=I+z(Q(t)-zI)^{-1},$$

we get

$$I_{(\varepsilon)}(f^{*})(t)=I'_{(\varepsilon)}(f^{*})(t)+I''_{(\varepsilon)}(f^{*})(t),$$

where

$$\begin{aligned}&I'_{(\varepsilon)}(f^{*})(t)\\&\quad=+\frac{1}{2i\pi}\int_{\gamma}(\mathrm{e}^{-\sqrt{-z}t}\sinh\sqrt{-z}(t-\varepsilon)\frac{Q(t)(Q(t)-zI)^{-1}}{z}f^{*}(t-\varepsilon)\mathrm{d}z\\&\qquad+\frac{1}{2i\pi}\int_{\gamma}\mathrm{e}^{-\sqrt{-z}(t+\varepsilon)}\cosh\sqrt{-z}t\frac{Q(t)(Q(t)-zI)^{-1}}{z}f^{*}(t+\varepsilon)\mathrm{d}z\\&\qquad+\frac{1}{2i\pi}\int_{\gamma}\left(\int_{0}^{t-\varepsilon}\frac{\mathrm{e}^{-\sqrt{-z}t}\sinh\sqrt{-z}s}{\sqrt{-z}}(Q(t)(Q(t)-zI)^{-1})f^{*}(s)\mathrm{d}s\right)\mathrm{d}z\\&\qquad+\frac{1}{2i\pi}\int_{\gamma}\left(\int_{t+\varepsilon}^{+\infty}\frac{\mathrm{e}^{-\sqrt{-z}s}\sinh\sqrt{-z}t}{\sqrt{-z}}(Q(t)(Q(t)-zI)^{-1})f^{*}(s)\mathrm{d}s\right)\mathrm{d}z,\end{aligned}$$

and

$$
\begin{aligned}
&I''_{(\varepsilon)}\left(f^*\right)(t)\\
&= -\frac{1}{2i\pi}\int_{\gamma_1}\int_0^{t-\varepsilon}\frac{\mathrm{e}^{-\sqrt{-z}t}\sinh\sqrt{-z}s}{\sqrt{-z}}f^*(s)\mathrm{d}s\mathrm{d}z\\
&\quad -\frac{1}{2i\pi}\int_{\gamma_1}\int_{t+\varepsilon}^{+\infty}\frac{\mathrm{e}^{-\sqrt{-z}s}\sinh\sqrt{-z}t}{\sqrt{-z}}f^*(s)\mathrm{d}s\mathrm{d}z\\
&\quad -\frac{1}{2i\pi}\int_{\gamma_1}\frac{\mathrm{e}^{-\sqrt{-z}t}\sinh\sqrt{-z}\,(t-\varepsilon)}{z}f^*(t-\varepsilon)\mathrm{d}z\\
&\quad -\frac{1}{2i\pi}\int_{\gamma_1}\frac{\mathrm{e}^{-\sqrt{-z}(t+\varepsilon)}\cosh\sqrt{-z}t}{z}f^*(t+\varepsilon)\mathrm{d}z.
\end{aligned}
$$

Now for the quantity $I'_{(\varepsilon)}\left(f^*\right)(t)$, one has

$$
\begin{aligned}
&I'_{(\varepsilon)}\left(f^*\right)(t)\\
&= +\frac{1}{2i\pi}\int_{\gamma_1}\mathrm{e}^{-\sqrt{-z}t}\sinh\sqrt{-z}\,(t-\varepsilon)\,\frac{Q\,(t)\,(Q\,(t)-zI)^{-1}}{z}f^*(t-\varepsilon)\mathrm{d}z\\
&\quad -\frac{1}{2i\pi}\int_{\gamma_1}\mathrm{e}^{-\sqrt{-z}(t+\varepsilon)}\sinh\sqrt{-z}t\,\frac{Q\,(t)\,(Q\,(t)-zI)^{-1}}{z}f^*(t)\mathrm{d}z\\
&\quad +\frac{1}{2i\pi}\int_{\gamma_1}\mathrm{e}^{-\sqrt{-z}(t+\varepsilon)}\cosh\sqrt{-z}t\,\frac{Q\,(t)\,(Q\,(t)-zI)^{-1}}{z}f^*(t+\varepsilon)\mathrm{d}z\\
&\quad -\frac{1}{2i\pi}\int_{\gamma_1}\mathrm{e}^{-\sqrt{-z}t}\cosh\sqrt{-z}\,(t-\varepsilon)\,\frac{Q\,(t)\,(Q\,(t)-zI)^{-1}}{z}f^*(t)\mathrm{d}z\\
&\quad +\frac{1}{2i\pi}\int_{\gamma_1}\left(\int_0^{t-\varepsilon}\frac{\mathrm{e}^{-\sqrt{-z}t}\sinh\sqrt{-z}s}{\sqrt{-z}}Q\,(t)\,(Q\,(t)-zI)^{-1}\left(f^*(s)-f^*(t)\right)ds\right)\mathrm{d}z\\
&\quad +\frac{1}{2i\pi}\int_{\gamma_1}\left(\int_{t+\varepsilon}^{+\infty}\frac{\mathrm{e}^{-\sqrt{-z}s}\sinh\sqrt{-z}t}{\sqrt{-z}}Q\,(t)\,(Q\,(t)-zI)^{-1}\left(f^*(s)-f^*(t)\right)\mathrm{d}s\right)\mathrm{d}z\\
&\quad +\frac{1}{2i\pi}\int_{\gamma_1}\mathrm{e}^{-\sqrt{-z}t}\,\frac{Q\,(t)\,(Q\,(t)-zI)^{-1}}{z}f^*(t)\mathrm{d}z.
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
&\lim_{\varepsilon\to 0} I'_{(\varepsilon)}\left(f^*\right)(t)\\
&= -\frac{1}{2i\pi}\int_{\gamma_1}\left(\int_0^{+\infty}k_{\sqrt{-z}}(t,s)\,Q\,(t)\,(Q\,(t)-z)^{-1}\left(f^*(s)-f^*(t)\right)\mathrm{d}s\right)\mathrm{d}z\\
&\quad +\frac{1}{2i\pi}\int_{\gamma_1}\mathrm{e}^{-\sqrt{-z}t}\,\frac{Q\,(t)\,(Q\,(t)-zI)^{-1}}{z}f^*(t)\mathrm{d}z.
\end{aligned}
$$

Concerning the term $I''_{(\varepsilon)}(f^*)(t)$, we write

$$
\begin{aligned}
& I''_{(\varepsilon)}\left(f^*\right)(t) \\
& = -\frac{1}{2i\pi}\int_{\gamma_1}\left(\int_0^{t-\varepsilon}\frac{\mathrm{e}^{-\sqrt{-z}t}\sinh\sqrt{-z}s}{\sqrt{-z}}\left(f^*(s)-f^*(t-\varepsilon)\right)\mathrm{d}s\right)\mathrm{d}z \\
& \quad -\frac{1}{2i\pi}\int_{\gamma_1}\frac{\mathrm{e}^{-\sqrt{-z}t}}{\sqrt{-z}}\int_0^{t-\varepsilon}\sinh\sqrt{-z}s ds f^*(t-\varepsilon)\mathrm{d}z \\
& \quad -\frac{1}{2i\pi}\int_{\gamma_1}\left(\int_{t+\varepsilon}^{+\infty}\frac{\mathrm{e}^{-\sqrt{-z}s}\sinh\sqrt{-z}t}{\sqrt{-z}}\left(f^*(s)-f^*(t+\varepsilon)\right)\mathrm{d}s\right)\mathrm{d}z \\
& \quad -\frac{1}{2i\pi}\int_{\gamma_1}\frac{\sinh\sqrt{-z}t}{\sqrt{-z}}\left(\int_{t+\varepsilon}^{+\infty}\mathrm{e}^{-\sqrt{-z}s})\mathrm{d}s\right)f^*(t+\varepsilon)\mathrm{d}z \\
& \quad -\frac{1}{2i\pi}\int_{\gamma_1}\frac{\mathrm{e}^{-\sqrt{-z}t}\sinh\sqrt{-z}\,(t-\varepsilon)}{z}f^*(t-\varepsilon)\mathrm{d}z \\
& \quad -\frac{1}{2i\pi}\int_{\gamma_1}\frac{\mathrm{e}^{-\sqrt{-z}(t+\varepsilon)}\cosh\sqrt{-z}t}{z}f^*(t+\varepsilon)\mathrm{d}z.
\end{aligned}
$$

A direct computation gives

$$\lim_{\varepsilon\to 0} I''_{(\varepsilon)}\left(f^*\right)(.) = 0.$$

On the other hand, one has

$$
\begin{aligned}
& Q(t)\,u(t) \\
& = -\frac{1}{2i\pi}\int_{\gamma_1}\int_0^{+\infty}k_{\sqrt{-z}}(t,s)Q(t)\left(Q(t)-zI\right)^{-1}\left(F^*(s)-f^*(t)\right)\mathrm{d}s\mathrm{d}z \\
& \quad -\frac{1}{2i\pi}\int_{\gamma_1}\frac{\mathrm{e}^{-\sqrt{-z}t}}{z}Q(t)\left(Q(t)-zI\right)^{-1}f^*(t)\mathrm{d}z \\
& \quad +\frac{1}{2i\pi}\int_{\gamma_1}\frac{1}{z}Q(t)\left(Q(t)-zI\right)^{-1}f^*(t)\mathrm{d}z.
\end{aligned}
$$

Summing up, we deduce that, for all $t \geq 0$:

$$
\begin{aligned}
& f(t) \\
& = \lim_{\varepsilon\to 0} u''_{\varepsilon}(t) + Q(t)\,u(t) \\
& = u''(t) + Q(t)\,u(t) \\
& = f^*(t) - Op\left(f^*\right)(t)\,.
\end{aligned}
$$

□

Now, we need the following important result concerning f^*.

Proposition 2.3 *There exists $\lambda^* > 0$, such that, for all $\lambda \geq \lambda^*$, the operator*

$$
\begin{aligned}
I - Op : BUC([0,+\infty[\,;E) &\to BUC([0,+\infty[\,;E) \\
f^*(.) &\mapsto f = f^*(.) - Op\left(f^*\right)(.)\,,
\end{aligned}
$$

is an isomorphism.

Proof It suffices to adapt the techniques used in [4, Proposition 5.2, p. 27]. □

To study the regularity of the formal solution, we need the following result

Proposition 2.4 *Assume that $f^* \in BUC^{2\sigma}([0,+\infty[\,;E)$. Then, the vector-valued function $t \mapsto Op\,(f^*)\,(t)$ belongs to the space $BUC^{2\sigma}([0,+\infty[\,;E)$.*

Proof Let $t > \tau \geq 0$, thus

$$Op\left(f^*\right)(t) - Op\left(f^*\right)(\tau) = (I) + (II),$$

where

$$\begin{aligned}
(\mathrm{I}) = {} & \frac{1}{2i\pi}\int_{\gamma}\left(\int_0^t \frac{e^{-\sqrt{-z}t}\sinh\sqrt{-z}s}{\sqrt{-z}}\frac{\partial^2}{\partial t^2}(Q(t)-zI)^{-1}f^*(s)\mathrm{d}s\right)\mathrm{d}z \\
& -\frac{1}{2i\pi}\int_{\gamma}\left(\int_0^\tau \frac{e^{-\sqrt{-z}\tau}\sinh\sqrt{-z}s}{\sqrt{-z}}\frac{\partial^2}{\partial \tau^2}(Q(\tau)-zI)^{-1}f^*(s)\mathrm{d}s\right)\mathrm{d}z \\
& +\frac{1}{2i\pi}\int_{\gamma}\left(\int_t^{+\infty} \frac{e^{-\sqrt{-z}s}\sinh\sqrt{-z}t}{\sqrt{-z}}\frac{\partial^2}{\partial t^2}(Q(t)-zI)^{-1}f^*(s)\mathrm{d}s\right)\mathrm{d}z \\
& -\frac{1}{2i\pi}\int_{\gamma}\left(\int_\tau^{+\infty} \frac{e^{-\sqrt{-z}s}\sinh\sqrt{-z}\tau}{\sqrt{-z}}\frac{\partial^2}{\partial \tau^2}(Q(\tau)-zI)^{-1}f^*(s)\mathrm{d}s\right)\mathrm{d}z,
\end{aligned}$$

and

$$\begin{aligned}
(\mathrm{II}) = {} & -\frac{1}{i\pi}\int_{\gamma}\left(\int_0^t e^{-\sqrt{-z}t}\sinh\sqrt{-z}s\,\frac{\partial}{\partial t}(Q(t)-zI)^{-1}f^*(s)\mathrm{d}s\right)\mathrm{d}z \\
& +\frac{1}{i\pi}\int_{\gamma}\left(\int_0^\tau e^{-\sqrt{-z}\tau}\sinh\sqrt{-z}s\,\frac{\partial}{\partial \tau}(Q(\tau)-zI)^{-1}f^*(s)\mathrm{d}s\right)\mathrm{d}z \\
& +\frac{1}{i\pi}\int_{\gamma}\left(\int_t^{+\infty} e^{-\sqrt{-z}s}\cosh\sqrt{-z}t\,\frac{\partial}{\partial t}(Q(t)-zI)^{-1}f^*(s)\mathrm{d}s\right)\mathrm{d}z \\
& -\frac{1}{i\pi}\int_{\gamma}\left(\int_\tau^{+\infty} e^{-\sqrt{-z}s}\cosh\sqrt{-z}\tau\,\frac{\partial}{\partial \tau}(Q(\tau)-zI)^{-1}f^*(s)\mathrm{d}s\right)\mathrm{d}z.
\end{aligned}$$

(I) and (II) can be treated similarly. So, we restrict ourselves to treat the first quantity. One has

$$(I) = (I_1) + (I_2),$$

where

$$\begin{aligned}
(I_1) = {} & +\frac{1}{2i\pi}\int_{\gamma}\int_0^\tau \frac{e^{-\sqrt{-z}t}\sinh\sqrt{-z}s}{\sqrt{-z}}\frac{\partial^2}{\partial t^2}(Q(t)-zI)^{-1}f^*(s)\mathrm{d}s\mathrm{d}z \\
& +\frac{1}{2i\pi}\int_{\gamma}\int_\tau^t \frac{e^{-\sqrt{-z}t}\sinh\sqrt{-z}s}{\sqrt{-z}}\frac{\partial^2}{\partial t^2}(Q(t)-zI)^{-1}f^*(s)\mathrm{d}s\mathrm{d}z
\end{aligned}$$

$$+\frac{1}{2i\pi}\int_{\gamma}\int_{0}^{\tau}\frac{e^{-\sqrt{-z}\tau}\sinh\sqrt{-z}s}{\sqrt{-z}}\frac{\partial^2}{\partial t^2}(Q(t)-zI)^{-1}f^*(s)\mathrm{d}s\mathrm{d}z$$
$$-\frac{1}{2i\pi}\int_{\gamma}\int_{0}^{\tau}\frac{e^{-\sqrt{-z}\tau}\sinh\sqrt{-z}s}{\sqrt{-z}}\frac{\partial^2}{\partial t^2}(Q(t)-zI)^{-1}f^*(s)\mathrm{d}s\mathrm{d}z$$
$$-\frac{1}{2i\pi}\int_{\gamma}\int_{0}^{\tau}\frac{e^{-\sqrt{-z}\tau}\sinh\sqrt{-z}s}{\sqrt{-z}}\frac{\partial^2}{\partial \tau^2}(Q(\tau)-zI)^{-1}f^*(s)\mathrm{d}s\mathrm{d}z,$$

and

$$(I_2)+\frac{1}{2i\pi}\int_{\gamma}\left(\int_{t}^{+\infty}\frac{e^{-\sqrt{-z}s}\sinh\sqrt{-z}t}{\sqrt{-z}}\frac{\partial^2}{\partial t^2}(Q(t)-zI)^{-1}f^*(s)\mathrm{d}s\right)\mathrm{d}z$$
$$-\frac{1}{2i\pi}\int_{\gamma}\left(\int_{t}^{+\infty}\frac{e^{-\sqrt{-z}s}\sinh\sqrt{-z}\tau}{\sqrt{-z}}\frac{\partial^2}{\partial \tau^2}(Q(\tau)-zI)^{-1}f^*(s)\mathrm{d}s\right)\mathrm{d}z,$$

We can write I_1 as

$$(I_1)=(J_1)+(J_2)+(J_3).$$

where

$$(J_1)=\frac{1}{2i\pi}\int_{\gamma_1}\left(\int_{0}^{\tau}\frac{\left(e^{-\sqrt{-z}t}-e^{-\sqrt{-z}\tau}\right)\sinh\sqrt{-z}s}{\sqrt{-z}}\frac{\partial^2}{\partial t^2}(Q(t)-zI)^{-1}f^*(s)\mathrm{d}s\right)\mathrm{d}z,$$
$$(J_2)=\frac{1}{2i\pi}\int_{\gamma_1}\left(\int_{0}^{\tau}\frac{e^{-\sqrt{-z}\tau}\sinh\sqrt{-z}s}{\sqrt{-z}}\left(\frac{\partial^2}{\partial t^2}(Q(t)-zI)^{-1}-\frac{\partial^2}{\partial \tau^2}(Q(\tau)-zI)^{-1}\right)f^*(s)\mathrm{d}s\right)\mathrm{d}z,$$
$$(J_3)=+\frac{1}{2i\pi}\int_{\gamma_1}\left(\int_{\tau}^{t}\frac{e^{-\sqrt{-z}t}\sinh\sqrt{-z}s}{\sqrt{-z}}\frac{\partial^2}{\partial t^2}(Q(t)-zI)^{-1}f^*(s)\mathrm{d}s\right)\mathrm{d}z,$$

thanks to the differentiability properties of the resolvent, that is, (5)and (6), we conclude that

$$\|(I_1)\|=O\,|t-\tau|^{2\sigma}.$$

The same arguments applied for I_2 give

$$\|(I_2)\|=O\,|t-\tau|^{2\sigma}.$$

□

Summing up, we are in position to give our main maximal regularity results concerning Problems (1) and (2).

Proposition 2.5 *Let $f\in BUC^{2\sigma}([0,+\infty[\,;E)$. Then, there exists $\lambda^*>0$, such that, for all $\lambda\geq\lambda^*$, Problems (1) and (2) have a unique strict solution*

$$u(t) = -\frac{1}{2i\pi} \int_{\gamma} \int_{0}^{+\infty} k_{\sqrt{-z}}(t, s) \left(Q(t) - z\right)^{-1} \left\{(I - Op)^{-1}(f)\right\}(s)\, \mathrm{d}s. \tag{11}$$

Moreover, one has

$$Q(.)u(.), u''(.) \in BUC^{2\sigma}([0, +\infty[\,; E).$$

Proof See Proposition 3.1 in [3]. □

The following lemma is needed to prove the optimal regularity of the strict solution (11) when f^* is taken in $L^\infty(]0, +\infty[\,; D_{Q(.)}(\sigma, +\infty)) \cap BUC([0, +\infty[\,, E)$ where

$$D_{Q(.)}(\sigma, +\infty) = \left\{\varphi \in E : \sup_{r>0} \left\| r^\sigma Q(.)\left(Q(.) - rI\right)^{-1}\varphi \right\|_E < +\infty\right\},$$

for more details about these spaces, see [7] and [13].

Lemma 2.6 *Let* $f^* \in L^\infty(]0, +\infty[\,; D_{Q(.)}(\sigma, +\infty)) \cap BUC([0, +\infty[\,, E)$*. Then, for all* $t \geq 0$*,*

$$u(t) \in D(A(t)).$$

Proof We are interesting with the convergence of the integral

$$\frac{1}{2i\pi} \int_{\gamma_1} \int_{0}^{+\infty} k_{\sqrt{-z}}(t, s) Q(t) \left(Q(t) - zI\right)^{-1} f^*(s) \mathrm{d}s \mathrm{d}z,$$

where $f^* \in L^\infty(]0, +\infty[\,; D_{Q(.)}(\sigma, +\infty))$.

One has

$$\left\| \int_{0}^{+\infty} k_{\sqrt{-z}}(t, s) Q(t) \left(Q(t) - zI\right)^{-1} f^*(s) \mathrm{d}s \right\|_E \leq (L_1) + (L_2),$$

with

$$\|(L_1)\| \leq C \frac{\mathrm{e}^{-(\Re\sqrt{-z})t}}{\left|\sqrt{-z}\right| |z|^\sigma} \frac{1}{\Re\sqrt{-z}} \sinh(\Re\sqrt{-z}t),$$

and

$$\|(L_2)\| \leq C \frac{\cosh(\Re\sqrt{-z}t)\mathrm{e}^{-(\Re\sqrt{-z})t}}{|z|^{1/2} |z|^\sigma\, \Re\sqrt{-z}},$$

so

$$\left\| \int_{0}^{+\infty} k_{\sqrt{-z}}(t, s) Q(t) \left(Q(t) - zI\right)^{-1} f^*(s) \mathrm{d}s \right\|_E \leq \frac{C}{|z|^{1+\sigma}}.$$ □

Therefore, we can deduce the following result.

Proposition 2.7 *Let* $f \in L^\infty(]0, +\infty[\,; D_{Q(.)}(\sigma, +\infty)) \cap BUC([0, +\infty[\,; E)$*. Then, there exists* $\lambda^* > 0$*, such that , for all* $\lambda \geq \lambda^*$*, Problems (1) and (2) have a unique strict solution given by (11). Moreover,*

$$Q(.)u(.), u''(.) \in L^\infty(]0, +\infty[\,; D_{Q(.)}(\sigma, +\infty)) \cap BUC([0, +\infty[\,; E).$$

3 On the study of a concrete elliptic problem set on nonsmooth cylindrical domain

3.1 Position of the problem

Now, we will apply the abstract regularity results obtained in the previous section for the study of a concrete elliptic problem. We consider the following problem:

Let Π be an open set of R^3 defined by

$$\Pi = \left\{(x_1, x_2, x_3) \in R^3 : (x_1, x_2) \in \Omega,\ 0 < x_3 < b\right\},$$

where b is a finite positive number and Ω is the planar cusp domain defined by

$$\Omega = \left\{(x_1, x_2) \in R^2 : 0 < x_1 < a,\ -\psi(x_1) < x_2 < \psi(x_1)\right\}.$$

Here

1. $\psi(x_1) = (x_1)^\alpha$, $(1 < \alpha \leq 2)$,
2. a is a finite positive number small enough.

In Π, we consider the boundary value problem

$$\begin{aligned} &\Delta u - \lambda \left(\tfrac{1}{\psi(x_1)}\right)^2 u = h\ ,\ \lambda > 0, \\ &u|_{\partial\Pi} = 0. \end{aligned} \tag{12}$$

The right-hand side h belongs to the Hölder space $C^{2\sigma}(\Pi)$ and satisfies the following condition

$$h = 0 \text{ on } \partial\Gamma_{(a)}. \tag{13}$$

where $\partial\Gamma_{(a)}$ denotes the the boundary of the lateral surface

$$\Gamma_{(a)} = \left\{(a, x_2, x_3) \in R^3 : -\psi(a) < x_2 < \psi(a)\ , 0 < x_3 < b\right\}.$$

It should be noted that Problem (12) is a particular case of some elliptic equations frequently encountered in engineering application. In fact, applications of such equation are abundant in fluid dynamics and the modelization of weather prediction.

It is well known that the solvability of elliptic problems posed in singular domains was intensively investigated by numerous authors via several techniques, see, for example [8,10]. Most of these studies have focused on the study of existence, uniqueness and the behavior of solutions near the singular parts of the boundary. In [5], an abstract approach was used to establish some Hölder continuous regularity results for the Dirichlet problem for Laplace equation posed in planar cusp domain. These authors have used the abstract differential equation theory which seems more adapted for this kind of problems. In our situation, the study of the concrete problem (12) will be reduced to the study of an abstract differential equation of elliptic type with variable operator coefficients., that is, Problems (1) and (2).

3.2 Change of variables

As in [5], we use the following change of variables

$$\begin{aligned} &T : \Pi \to \Pi_\infty \\ &(x_1, x_2, x_3) \mapsto (\xi, \eta, \upsilon) := \left(\tfrac{1}{\alpha-1}(x_1)^{1-\alpha},\ \tfrac{x_2}{\psi(x_1)}, x_3\right), \end{aligned}$$

with

$$\Pi_\infty =]\xi_0, +\infty[\times D,\ \xi_0 = \frac{1}{\alpha - 1}(a)^{1-\alpha} > 0 \quad \text{and} \quad D =]-1, 1[\times]0, b[\ .$$

which means that the cuspidal edge $(0, 0, x_3)$, $(0 < x_3 < b)$ is transformed in

$$D_\infty = \{+\infty\} \times D.$$

Note that for any $x_1 \in]0, a[, x_3 \in]0, b[$

$$\begin{cases} \lim\limits_{x\to 0^+} T(x_1, \psi(x_1), x_3) = \lim\limits_{x\to 0^+} \left(\frac{1}{\alpha-1}(x_1)^{1-\alpha}, 1, \upsilon\right) = (+\infty, 1, \upsilon) \\ \lim\limits_{x\to 0^+} T(x_1, -\psi(x_1), x_3) = \lim\limits_{x\to 0^+} \left(\frac{1}{\alpha-1}(x_1)^{1-\alpha}, -1, \upsilon\right) = (+\infty, -1, \upsilon). \end{cases}$$

Now, define the following change of functions

$$\begin{cases} v(\xi, \eta, \upsilon) := u(x_1, x_2, x_3) = u\left(\exp\left(\frac{\ln[(\alpha-1)\xi]}{1-\alpha}\right), \eta \exp\left(\frac{\alpha \ln[(\alpha-1)\xi]}{1-\alpha}\right), \upsilon\right), \\ g(\xi, \eta, \upsilon) := h(x_1, x_2, x_3) = h\left(\exp\left(\frac{\ln[(\alpha-1)\xi]}{1-\alpha}\right), \eta \exp\left(\frac{\alpha \ln[(\alpha-1)\xi]}{1-\alpha}\right), \upsilon\right). \end{cases}$$

Remark 3.1 Observe that if h is continuous in $(0, 0, x_3)$, one has

$$\begin{aligned} &\lim_{\xi\to+\infty} g(\xi, \eta, \upsilon) \\ &= \lim_{\xi\to+\infty} h\left(\exp\left(\frac{\ln[(\alpha-1)\xi]}{1-\alpha}\right), \eta \exp\left(\frac{\alpha \ln[(\alpha-1)\xi]}{1-\alpha}\right), \upsilon\right) \\ &= h(0, 0, x_3). \end{aligned}$$

From the above changes of variables, one has

$$\begin{aligned} &\Delta u \\ &= \theta^2\xi^{2\beta}\Bigg(\Delta_{(\xi,\eta)} v + \theta^{-2}\xi^{-2\beta}\partial_\upsilon^2 v + \alpha^2\theta^{-2/\beta}\left(\frac{\eta}{\xi}\right)^2 \partial_\eta^2 + 2\alpha\theta^{-1/\beta}\frac{\eta}{\xi}\partial_{\xi\eta}^2 \\ &+\alpha\theta^{-1/\beta}\left(\frac{1}{\xi}\right)\partial_\xi v + \alpha(\alpha+1)\theta^{-2/\beta}\eta\left(\frac{1}{\xi}\right)^2 \partial_\eta^2 v\Bigg) \end{aligned}$$

It follows that Problem (12) becomes

$$\begin{cases} \theta^{-2}\xi^{-2\beta}\partial_\upsilon^2 v + \Delta_{(\xi,\eta)} v - \lambda v + \frac{1}{\xi}[Pv] = f & (\xi, \eta, \upsilon) \in Q \\ v(\xi, \eta, \upsilon) = 0 & (\xi, \eta, \upsilon) \in \partial Q, \end{cases} \tag{14}$$

with

$$f(\xi, \eta, \upsilon) = \theta^{-2}\xi^{-2\beta} g(\xi, \eta, \upsilon), \ (\xi, \eta, \upsilon) \in \Pi_\infty,$$

here,

$$\beta = \alpha/(\alpha-1), \theta = (\alpha-1)^\beta \quad \text{and} \quad \Delta_{(\xi,\eta)} = \partial_\xi^2 + \partial_\eta^2.$$

We have also

$$f(\xi_0, \eta, \upsilon) = 0 \quad \text{on } \partial D, \tag{15}$$

here P is the second-order differential operator with C^∞ -bounded coefficients on Π_∞ given by

$$\begin{aligned} &[P(\xi, \eta, \upsilon) v](\xi, \eta, \upsilon) \\ &= \alpha^2\theta^{-2/\beta}\eta^2\left(\frac{1}{\xi}\right)\partial_\eta^2 v(\xi, \eta, \upsilon) + 2\alpha\theta^{-1/\beta}\eta\partial_{\xi\eta}^2 v(\xi, \eta, \upsilon) \\ &+\alpha\theta^{-1/\beta}\partial_\xi v(\xi, \eta, \upsilon) + \alpha(\alpha+1)\theta^{-2/\beta}\eta\left(\frac{1}{\xi}\right)\partial_\eta v(\xi, \eta, \upsilon). \end{aligned}$$

In the sequel, we will focus ourselves on the study of the concrete problem

$$\begin{cases} \xi^{-2\beta}\partial_\upsilon^2 v + \Delta_{(\xi,\eta)} v - \lambda v = f & (\xi, \eta, \upsilon) \in \Pi_\infty \\ v(\xi, \eta, \upsilon) = 0 & (\xi, \eta, \upsilon) \in \partial\Pi_\infty, \end{cases} \tag{16}$$

Remark 3.2 Observe that the right-hand side f, is necessarily bounded on Π_∞ , since

$$\begin{aligned}|f(\xi,\eta,\upsilon)| &= \left|\theta^{-2}\xi^{-2\beta}h\left(\exp\left(\frac{\ln[(\alpha-1)\xi]}{1-\alpha}\right),\eta\exp\left(\frac{\alpha\ln[(\alpha-1)\xi]}{1-\alpha}\right),\upsilon\right)\right| \\ &\le \theta^{-2}(\xi_0)^{-2\beta}\left|h\left(\exp\left(\frac{\ln[(\alpha-1)\xi]}{1-\alpha}\right),\eta\exp\left(\frac{\alpha\ln[(\alpha-1)\xi]}{1-\alpha}\right),\upsilon\right)\right| \\ &\le C\max_{(x_1,x_2,x_3)\in\Pi}|h(x_1,x_2,x_3)|.\end{aligned}$$

Remark also that our change of variables leads to the following property on the behavior on f at $+\infty$. For all $(\eta,\upsilon)\in D$

$$\begin{aligned}&\lim_{\xi\to+\infty} f(\xi,\eta,\upsilon) \\ &= \theta^{-2}\lim_{\xi\to+\infty}\xi^{-2\beta}h\left(\exp\left(\frac{\ln[(\alpha-1)\xi]}{1-\alpha}\right),\eta\exp\left(\frac{\alpha\ln[(\alpha-1)\xi]}{1-\alpha}\right),\upsilon\right) \\ &= \theta^{-2}\lim_{\xi\to+\infty}\xi^{-2\beta}h(0,0,x_3) \\ &= 0.h(0,0,x_3) \\ &= 0.\end{aligned}$$

Using the same arguments as in [5, Proposition 3.1], we get

Lemma 3.3 *Let* $0<2\sigma<1$. *Then*

1. $h\in C^{2\sigma}(\Pi)\Rightarrow g\in BUC^{2\sigma}(\Pi_\infty)$.
2. $g\in BUC^{2\sigma}(\Pi_\infty)\Rightarrow (x_1)^{4\sigma\alpha}h\in C^{2\sigma}(\Pi)$.

3.3 The abstract formulation of Problem (16)

Set $E=BUC\left(\overline{D}\right)$. Define the vector-valued following functions:

$$\begin{aligned}&\upsilon:[\xi_0,+\infty[\to E;\quad \xi\longrightarrow\upsilon(\xi);\quad \upsilon(\xi)(\eta,\upsilon)=\upsilon(\xi,\eta,\upsilon),\\ &f:[\xi_0,+\infty[\to E;\quad \xi\longrightarrow f(\xi);\quad f(\xi)(\eta,\upsilon)=f(\xi,\eta,\upsilon).\end{aligned}$$

Consider the family of closed linear operators $(A(\xi))_{\xi\ge 0}$ defined by

$$\begin{cases} D(A(\xi))=\left\{\varphi\in BUC_0\left(\overline{D}\right)\cap W^{2,q}(D),\, q>2,\ A(\xi)\varphi\in BUC\left(\overline{D}\right)\right\},\\ A(\xi)\varphi(\eta,\upsilon)=\left(\partial_\eta^2+\xi^{-2\beta}\partial_\upsilon^2\right)\varphi,\ \xi\ge\xi_0.\end{cases}$$

where

$$BUC_0\left(\overline{D}\right)=\left\{\phi\in BUC\left(\overline{D}\right)/\phi=0\ \text{ on }\ \partial D\right\}.$$

Then, the concrete problem (16) is written in the following operational form

$$\begin{cases}\upsilon''(\xi)+A(\xi)\upsilon(\xi)-\lambda\upsilon(\xi)=f(\xi), & \xi\ge\xi_0,\\ \upsilon(\xi_0)=0,\,\upsilon(+\infty)=0,\end{cases}\tag{17}$$

where

$$f\in L^\infty(]\xi_0,+\infty[\,;BUC^{2\sigma}\left(\overline{D}\right))\cap BUC^{2\sigma}([\xi_0,+\infty[\,;BUC(\overline{D})).$$

It will be more convenient to work on $[0,+\infty[$ instead of $[\xi_0,+\infty[$. So, we consider the natural change of function: for $\xi\in[0,+\infty[$, set

$$V(\xi)=\upsilon(\xi+\xi_0),\quad F(\xi)=f(\xi+\xi_0).$$

Therefore, it is clear that

$$F \in L^{\infty}(]0,+\infty[\,; BUC^{2\sigma}\left(\overline{D}\right)) \cap BUC^{2\sigma}([0,+\infty[\,; BUC(\overline{D})).$$

Now the complete analysis of (17) on $[\xi_0, +\infty[$ is equivalent to the one done for the following problem

$$\begin{cases} V''(\xi) + A_0(\xi) V(\xi) - \lambda V(\xi) = F(\xi) & \xi \geq 0, \\ V(0) = 0, V(+\infty) = 0. \end{cases} \tag{18}$$

where

$$\begin{cases} D(A_0(\xi)) = \left\{\varphi \in BUC_0\left(\overline{D}\right) \cap W^{2,q}(D), q > 2,\ A_0(\xi)\varphi \in BUC\left(\overline{D}\right)\right\}, \\ A_0(\xi)\varphi(\eta, \upsilon) = \left(\partial_\eta^2 + (\xi+\xi_0)^{-2\beta}\,\partial_\upsilon^2\right)\varphi,\ \ \xi \geq 0. \end{cases} \tag{19}$$

For simplicity, in the sequel , we write (18) as

$$\begin{cases} V''(\xi) + Q(\xi) V(\xi) = F(\xi) & \xi \geq 0, \\ V(0) = 0, V(+\infty) = 0. \end{cases}$$

where

$$Q(\xi) := A_0(\xi) - \lambda,\ \lambda > 0.$$

3.4 Optimal results for Problem (12)

At this level, it is important to recall that the spectral properties of the family (19) in its most general form were deeply discussed in [1], [12] and [14]. From which, we can deduce that

Lemma 3.4 *$(A(\xi))_{\xi\geq 0}$ is a family of closed linear operators with non-dense domains in E verifying the assumptions (3)–(6).*

Remark 3.5 Observe that in our situation thanks to (13), it is well known that

$$D_{A(.)}(\sigma, +\infty) = \left\{\phi \in BUC^{2\sigma}(D) : \phi = 0 \ \text{ on } \ \partial D\right\}. \tag{20}$$

For more details, see [13].

Then, our abstract results (2.5) and (2.6) applied for Problem (16) give

Proposition 3.6 *Let $f \in BUC^{2\sigma}(\Pi_\infty)$, $0 < 2\sigma < 1$. Then, there exists $\lambda^* > 0$ such that for all $\lambda \geq \lambda^*$, the problem*

$$\begin{cases} \xi^{-2\beta}\partial_\upsilon^2 v + \Delta_{(\xi,\eta)} v - \lambda v = f, & (\xi,\eta,\upsilon) \in \Pi_\infty, \\ v(\xi,\eta,\upsilon) = 0, & (\xi,\eta,\upsilon) \in \partial\Pi_\infty. \end{cases}$$

admits a unique strict solution w satisfying

$$\partial_\xi^2 v,\ \left(\partial_\eta^2 + \xi^{-2\beta}\partial_\upsilon^2\right) v \in BUC^{2\sigma}(\Pi_\infty).$$

Now, adapting the same argument of perturbation used in [2] and [9], we are able to say that

Proposition 3.7 *Let $k \in BUC^{2\sigma}(\Pi_\infty)$, $0 < 2\sigma < 1$. Then, there exist $\lambda^* > 0$ and $\xi^* > \xi_0$ such that for all $\lambda \geq \lambda^*$ and $\xi \geq \xi^*$, the problem*

$$\begin{cases} \xi^{-2\beta}\partial_\upsilon^2 v + \Delta_{(\xi,\eta)} v - \lambda v + \frac{1}{\xi} P v = f, & (\xi,\eta,\upsilon) \in \Pi_\infty, \\ v(\xi,\eta,\upsilon) = 0, & (\xi,\eta,\upsilon) \in \partial\Pi_\infty. \end{cases}$$

admits a unique strict solution v satisfying

$$\partial_\xi^2 v,\ \left(\partial_\eta^2 + \xi^{-2\beta}\partial_\upsilon^2\right) v \in BUC^{2\sigma}(\Pi_{\infty\,\xi^*}).$$

where

$$\Pi_{\infty\,\xi^*} = \left[\xi^*, +\infty\right[\times D$$

Let $x_1^* := T^{-1}(\xi^*)$ where T^{-1} is the inverse change of variables given by

$$\begin{array}{l} T^{-1} : \Pi_\infty \to \Pi \\ (\xi, \eta, \upsilon) \mapsto (x_1, x_2, x_3) \end{array} \tag{21}$$

with

$$(x_1, x_2, x_3) := \left(\exp\left(\frac{\ln[(\alpha-1)\xi]}{1-\alpha}\right), \eta\exp\left(\frac{\alpha\ln[(\alpha-1)\xi]}{1-\alpha}\right), \upsilon\right)$$

First, it is easy to see that

$$x_1^* < x_1.$$

Using (21), we are then in position to state our main result describing the regularity of the unique solution u of Problem (12) near the cuspidal edge

Theorem 3.8 *Let $h \in C^{2\sigma}(\Pi)$, $0 < 2\sigma < 1$, satisfying (13). Then, there exist $x_1^* > 0$ and $\lambda^* > 0$ such that, for all $\lambda > \lambda^*$ Problem (12) admits a unique strict solution u satisfying*

$$(x_1)^{4\sigma\alpha}\,\partial_{x_1}^2 u \text{ and } (x_1)^{4\sigma\alpha}\left(\Delta_{(x_2,x_3)} - \lambda\right)u \in C^{2\sigma}(\Pi_{x_1^*}),$$

where

$$\Pi_{x_1^*} = \left\{(x_1, x_2, x_3) \in R^3 : (x_1, x_2) \in \Omega_{x_1^*}, 0 < x_3 < b\right\},$$

with

$$\Omega_{x_1^*} = \left\{(x_1, x_2) \in R^2 : 0 < x_1 < x_1^*, -(x_1)^\alpha < x_2 < (x_1)^\alpha\right\}.$$

Acknowledgments The authors would like to thank the referees for their interesting comments and suggestions.

References

1. Acquistapace, P.; Terreni, B.: Some existence and regularity results for abstract non-autonomous parabolic equations. J. Math. Anal. Appl. **99**(1), 9–64 (1984)
2. Belahdji, K.: La régularité L^p de la solution du problème de Dirichlet dans un domaine à points de rebroussement. C. R. Acad. Sci. Paris Sér. I **322**, 5–8 (1996)
3. Berroug, T.; Labbas, R.; Sadallah, B.K.: Resolution in Hölder spaces of an elliptic problem in an unbounded domain. J. Aust. Math. Soc. **8**(1), 387–404 (2006)
4. Boutaous, F.; Labbas, R.; Sadallah, B.K.: Fractional-power approach for solving complete elliptic abstract differential equations with variable-operator coefficients. Electron. J. Differ. Equ. **2012**(05), 1–33 (2012)
5. Chaouchi, B.; Labbas, R.; Sadallah, B.K.: Laplace equation on a domain with a cuspidal point in Little Hölder spaces, Mediterr. J. Math. **10**(1), 157–175 (2013)
6. Da Prato, G.; Grisvard, P.: Sommes d'operateurs lineaires et equations differentielles operationnelles. J. Math. Pures Appl. **54**, 305–387 (1975)
7. Grisvard, P.: Spazi di tracce e applicazioni. Rend. Mat. **5**, 657–729 (1972)
8. Grisvard, P.: Elliptic problems in nonsmooth domains. In: Monographs and Studies in Mathematics, vol. 24. Pitman Advanced Publishing Program, Pitman Publishing Inc., Boston (1985)
9. Grisvard, P.: Problèmes aux limites dans des domaines avec points de rebroussement. In: Partial differential equations and functional analysis, progress in nonlinear differential equations applications, vol. 22. Birkhäuser, Boston (1996)
10. Kozlov, V.A.; Maz'ya, V.G.; Rossmann, J.: Elliptic boundary value problems in domains with point singularities. In: Mathematical Surveys and Monographs (1997)
11. Krein, S.G.: Linear differential equations in Banach space. In: Translations of Mathematical Monographs, vol. 29 (1971)
12. Labbas, R.: Problèmes aux Limites pour une Equation Différentielle Abstraite du Second Ordre. Thèse d'état, Université de Nice, Juin (1987)
13. Lunardi, A.: Analytic Semigroups and Optimal Regularity in Parabolic Problems. Birkhäuser, Boston (1995)
14. Stewart, H.B.: Generation of analytic semi group by strongly elliptic operators under general boundary conditions. Trans. Am. Math. Soc. **259**, 299–310 (1980)
15. Tanabe, H.: Equations of evolution. In: Monographs and Studies in Mathematics, vol. 6. Pitman, London (1979)
16. Yakubov, S.; Yakubov, Y.: Differential-operator equations. Ordinary and partial differential equations. In: Monographs and Surveys in Pure and Applied Mathematics, vol. 103. Chapman & Hall/CRC, Florida (2000)